

Fractional Langevin Equation: Over-Damped, Under-Damped and Critical Behaviors.

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The dynamical phase diagram of the fractional Langevin equation is investigated for harmonically bound particle. It is shown that critical exponents mark dynamical transitions in the behavior of the system. Four different critical exponents are found. (i) $\alpha_c = 0.402 \pm 0.002$ marks a transition to a non-monotonic under-damped phase, (ii) $\alpha_R = 0.441\dots$ marks a transition to a resonance phase when an external oscillating field drives the system, (iii) $\alpha_{\chi_1} = 0.527\dots$ and (iv) $\alpha_{\chi_2} = 0.707\dots$ marks transition to a double peak phase of the “loss” when such an oscillating field present. As a physical explanation we present a cage effect, where the medium induces an elastic type of friction. Phase diagrams describing over-damped, under-damped regimes, motion and resonances, show behaviors different from normal.

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I. INTRODUCTION

In this paper we investigate the phenomenological description of stochastic processes using a Fractional Langevin Equation (FLE) [1, 2, 3, 4, 5, 6, 7]. While in simple systems the memory friction kernel is an exponentially decaying function or a delta function, in complex out of equilibrium systems the picture is in some cases different. Namely the relaxation is of a power law type, and the particle may exhibit anomalous diffusion and relaxation [8]. Mathematically such systems are modeled using fractional calculus, e.g. $\frac{d^{1/2}}{dt^{1/2}}$. An example is the recent experiment on protein dynamics of the group of Xie [9, 10]. There anomalous dynamics of the coordinate x , describing donor-acceptor distance was recorded, and a FLE (see Sec. II) was found to describe the experimental data. The motion of x is bounded by a harmonic force field, and the equation of motion for the average $\langle x \rangle$ is

$$\langle \ddot{x} \rangle + \omega^2 \langle x \rangle + \gamma \frac{d^\alpha \langle x \rangle}{dt^\alpha} = 0 \quad (1)$$

where $0 < \alpha < 1$, ω is the harmonic frequency and $\gamma > 0$. For the case $\alpha = 1$ we get the usual damped oscillator [11]. For such a normal case two types of behaviors: the under-damped and the over-damped motions are found. In the under-damped case $\langle x \rangle$ is oscillating, and crossing the zero line, while for over-damped case $\langle x \rangle$ is monotonically decaying with no zero crossing. For $\alpha = 1$, there exist a critical frequency $\omega_c = \frac{\gamma}{2}$ which separates these two types of motion. Here we explore a similar scenario for the fractional oscillator, and find rich types of physical behaviors. It is known [12] that in the long time limit all solutions $\langle x \rangle$ (i.e. any $0 < \alpha$, $0 < \gamma$ and $0 < \omega$) decay monotonically, some what like the over-damped behavior of the usual oscillator, however now the decay is of power law type. The interesting Physics occurs at shorter times where the solution may exhibit different types of relaxation and oscillations. We find that for $\alpha < \alpha_c$ the solution is non-monotonic for any set of parameters ($\gamma, \omega > 0$). Thus we find a critical α which marks a dynamical transition in the behavior of the system.

We also investigate the response of a system described by Eq. (1) to an external oscillating force $F_0 \cos(\Omega t)$. For the regular case of $\alpha = 1$, a resonance is present if the frequency ω is larger than the critical value $\gamma/\sqrt{2}$. The behavior for $0 < \alpha < 1$ is quite different and we discover that the transition between $\alpha \rightarrow 1$ and $\alpha \rightarrow 0$ is not smooth. In particular we find another critical exponent α_R , where for $\alpha < \alpha_R$ a resonance is always present. Other critical exponents are found for the imaginary part of complex susceptibility. Our goal is to clarify the nature of solution to Eq. (1), investigate the meaning of fractional critical frequency with and without external oscillating force and provide a mathematical tool box for finding and plotting solutions of Eq. (1). Our finding that critical α s mark dynamical transitions is very surprising and couldn't be obtained without our mathematical treatment.

The paper is organized as follows. In Sec. II we present the FLE. In Sec. III we present two different methods for solution for first order moments and examples are solved in Sec. IV. In Sec. V we present different definitions for over-damped and under-damped motion and find the critical exponent α_c . We also interpret our results from a more Physical point of view and discuss the cage effect as a viscoelastic property of the medium. In Sec. VI we introduce external oscillating force into the system and find the response for such force for the free (Sec. VIA) and harmonically bounded (Sec. VIB) particles. In Sec. VIC we investigate the properties of the “loss” - the imaginary part of complex susceptibility. A summary is provided in Sec. VII, and the three Appendixes deal with some technical aspects. A brief summary of some of our results was published [13].

II. THE MODEL

We consider the FLE

$$m \frac{d^2 x(t)}{dt^2} + \bar{\gamma} \int_0^t \frac{1}{(t-\bar{t})^\alpha} \frac{dx}{d\bar{t}} d\bar{t} = F(x, t) + \xi(t) \quad (2)$$

where $\bar{\gamma} > 0$ is a generalized friction constant ($\gamma = \frac{1}{m}\bar{\gamma}\Gamma(1-\alpha)$), $0 < \alpha < 1$ is the fractional exponent and $\xi(t)$ is a stationary, Fractional Gaussian noise [14, 15] satisfying the fluctuation-dissipation relation [16]

$$\langle \xi(t) \rangle = 0, \quad \langle \xi(t)\xi(\tilde{t}) \rangle = k_b T \bar{\gamma} |t - \tilde{t}|^{-\alpha}. \quad (3)$$

$F(x, t)$ is an external force. We follow experiment [10] and assume $F(x, t) = -m\omega^2 x$, later in Sec. VI we will have $F(x, t) = -m\omega^2 x + F_0 \cos(\Omega t)$. In Laplace Space it is easy to show, using the convolution theorem

$$\begin{aligned} \hat{x}(s) = & \frac{s + \frac{1}{m}\beta(s)}{s^2 + \frac{1}{m}s\beta(s) + \omega^2} x_0 + \frac{1}{s^2 + \frac{1}{m}s\beta(s) + \omega^2} v_0 \\ & + \frac{1}{s^2 + \frac{1}{m}s\beta(s) + \omega^2} \xi(s) \end{aligned} \quad (4)$$

where x_0 and v_0 are initial conditions and

$$\beta(s) = \bar{\gamma}\Gamma(1-\alpha)s^{\alpha-1}. \quad (5)$$

All along this work the variable in the parenthesis defines the space we are working in (e.g. $\hat{x}(s)$ is the Laplace Transform of $x(t)$). Eq. (2) with $\alpha = \frac{1}{2}$ and $F(x, t) = -m\omega^2 x$, describes single protein dynamics [10]. An experimentally measured quantity is the normalized correlation function

$$C_x(t) = \frac{\langle x(t)x(0) \rangle}{\langle x(0)^2 \rangle}. \quad (6)$$

In what follows, thermal initial conditions are assumed $\langle \xi(t)x(0) \rangle = 0$, $\langle x(0)^2 \rangle = k_b T / m\omega^2$ and $\langle x(0)v(0) \rangle = 0$. From Eq. (4) we find

$$\hat{C}_x(s) = \frac{s + \gamma s^{\alpha-1}}{s^2 + \gamma s^{\alpha} + \omega^2}. \quad (7)$$

It is easy to show that $C_x(t)$ satisfies the following fractional Eq.

$$\ddot{C}_x(t) + \omega^2 C_x(t) + \gamma \frac{d^\alpha C_x(t)}{dt^\alpha} = 0, \quad (8)$$

with the initial conditions $C_x(0) = 1$ and $\dot{C}_x(0) = 0$, where the fractional derivative is defined in the Caputo sense [17, 18]

$$\frac{d^\alpha f(t)}{dt^\alpha} = {}_0D_t^{\alpha-1} \left(\frac{df(t)}{dt} \right) \quad (9)$$

and ${}_0D_t^{\alpha-1}$ is the Riemann-Liouville fractional operator [17, 18]

$${}_0D_t^{\alpha-1} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tilde{t})^{-\alpha} f(\tilde{t}) d\tilde{t}. \quad (10)$$

Note that another way to write Eq. (2) is

$$\ddot{x} + \gamma \frac{d^\alpha x}{dt^\alpha} + \omega^2 x = \xi(t) \quad (11)$$

hence the name Fractional Langevin equation is justified. For the force free particle ($F(x, t) = 0$ in Eq. (2)) $\langle x^2 \rangle \propto t^\alpha$ [6, 19], this is a sub-diffusive behavior since $0 < \alpha < 1$.

The FLE in general and Eq. (8) in particular can be derived from the Kac-Zwanzig model of a Brownian particle coupled to a Harmonic bath [20]. Eq. (8) can be derived also from the Fractional Kramers Equation [21]. The use of fractional-differential equations like Eqs. (8,11) became quite common in recent years [8, 22], especially in the context of anomalous diffusion [8, 23, 24]. Several other fractional oscillator equations were considered in the literature [25, 26, 27] and general solutions for fractional-differential equations of the type Eq. (8) are studied in the mathematical literature [18, 28, 29]. In the next section we present a practical recipe, a tool box, for the solution of Eq. (8), and show how to plot its solution. Our methods are general and can be applied to other linear fractional differential equations. From a more physical point of view the following questions are addressed in the next sections: (i) When is $C_x(t)$ positive? (i.e. similar to over-damped behavior, $\alpha = 1$). (ii) When is $C_x(t)$ non-monotonic? (similar to under-damped motion for $\alpha = 1$) (iii) What is the analogue to the critical frequency $\omega_c = \frac{\gamma}{2}$ found for the $\alpha = 1$ case. (iv) Does a critical exponent α_c exist and if so, what is its value?

III. THE GENERAL SOLUTION

In this section we will present a recipe for an analytical solution of Eq. (8).

The formula for $C_x(t)$ in Laplace space is Eq. (7) and if α was an integer then performing an Inverse Laplace Transform would be simple using analysis of poles [30, 31], because then the denominator of Eq. (7) is a polynomial. We assume α is of the form $\alpha = \frac{p}{q}$, where $q > p > 0$ are integers and $\frac{p}{q}$ is irreducible (i.e not equal to some other $\frac{l}{n}$ where $l < p$ and $n < q$ are integers).

A. Method A

We rewrite Eq. (7) as

$$\hat{C}_x(s) = \frac{s + \gamma s^{\frac{p}{q}-1}}{s^2 + \gamma s^{\frac{p}{q}} + \omega^2} \frac{\hat{Q}(s)}{\hat{Q}(s)} = \frac{(s + \gamma s^{\frac{p}{q}-1}) \hat{Q}(s)}{\hat{P}(s)} \quad (12)$$

where $\hat{P}(s)$ is a polynomial in s . According to a mathematical theorem [18] we can always find $\hat{Q}(s)$ such that the denominator of Eq. (12)

$$\hat{P}(s) = (s^2 + \gamma s^{\frac{p}{q}} + \omega^2) \hat{Q}(s) \quad (13)$$

is a regular polynomial in s . The polynomial $\hat{Q}(s)$ is called the complementary polynomial. The task of finding $\hat{Q}(s)$ is simple, $\hat{Q}(s)$ is a polynomial in $s^{\frac{1}{q}}$ of degree

$2q(q-1)$

$$\hat{Q}(s) = \sum_{m=0}^{2q(q-1)} B_m s^{\frac{m}{q}} \quad (14)$$

with $B_{2q(q-1)} = 1$. The coefficients B_m are found by equating all the coefficients in the expansion of the product $\hat{Q}(s)(s^2 + \gamma s^{\frac{p}{q}} + \omega^2)$, with non-integer powers of s , to zero. This produces a linear system of $2q(q-1)$ equations for $2q(q-1)$ variables, which in principle is a solvable problem.

We also assume that all $2q$ zeros of $\hat{P}(s)$ are distinct. The generalization to the case where the zeros are not distinct will be treated later, such a behavior is related to a critical frequency of the system. We use the partial fraction expansion

$$\frac{1}{\hat{P}(s)} = \sum_{k=1}^{2q} \frac{A_k}{s - a_k} \quad (15)$$

where a_k are the solutions of $\hat{P}(s) = 0$ and A_k are constants defined as

$$A_k = \frac{1}{\left. \frac{d\hat{P}(s)}{ds} \right|_{s=a_k}}. \quad (16)$$

It can be shown [18] that

$$\frac{s^m}{\hat{P}(s)} = \sum_{k=1}^{2q} \frac{a_k^m A_k}{s - a_k}, \quad m = 0, 1, \dots, 2q-1. \quad (17)$$

The numerator of Eq. (12) is written using the expansion

$$\hat{Q}(s) \left(s + \gamma s^{\frac{p}{q}-1} \right) = \sum_{m=0}^{2q-1} \sum_{j=0}^{q-1} \tilde{B}_{mj} s^{m-\frac{j}{q}}. \quad (18)$$

Hence using Eqs. (12,15,18)

$$\hat{C}_x(s) = \sum_{m=0}^{2q-1} \sum_{j=0}^{q-1} \sum_{k=1}^{2q} \frac{a_k^m \tilde{B}_{mj} A_k}{s - a_k} s^{-\frac{j}{q}}. \quad (19)$$

So finally we reduced the problem of calculating the Inverse Laplace Transform of Eq. (7) to performing Inverse Laplace Transform for

$$\frac{1}{s - a_k} \bullet \circ e^{a_k t} \quad (20)$$

and

$$\frac{1}{s^{\frac{j}{q}}(s - a_k)} \bullet \circ \frac{e^{a_k t}}{\Gamma(\frac{j}{q} a_k^{\frac{j}{q}})} \gamma(\frac{j}{q}, a_k t) \quad (21)$$

where $\gamma(\frac{j}{q}, a_k t)$ is a tabulated Incomplete Gamma Function [32]. Using the series expansion for $\gamma(a, x)$

$$\gamma(a, x) = \Gamma(a) x^a e^{-x} \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(a+n+1)}$$

we can write Eq. (21) by the means of generalized Mittag-Leffler function [33]

$$\frac{1}{s^{\frac{j}{q}}(s - a_k)} \bullet \circ t^{\frac{j}{q}} E_{1, 1+\frac{j}{q}}(a_k t). \quad (22)$$

The Mittag-Leffler function satisfies

$$E_{\eta, \mu}(y) = \sum_{n=0}^{\infty} \frac{y^n}{\Gamma(\eta n + \mu)} \quad (23)$$

with

$$E_{\eta, \mu}(y) \sim -\frac{y^{-1}}{\Gamma(\mu - \eta)} \quad y \rightarrow \infty. \quad (24)$$

Actually we have finished our task, we can now perform an Inverse Laplace Transform of expressions like Eq. (7) and even more general expressions which can be presented as a fraction of polynomials with fractional powers. To summarize, we need to follow four steps (i) Calculate $\hat{Q}(s)$, which is equivalent to the diagonalisation of a matrix of size $(2q(q-1))^2$. (ii) Find the zeros of $\hat{P}(s)$, a_k , and the coefficients of partial fractions expansion, A_k of Eq. (16). (iii) Find the coefficients \tilde{B}_{mj} for Eq. (18) and write $\hat{C}_x(s)$ as the sum in Eq. (19). (iv) Use Eqs. (20,21,22) to inverse Laplace transform Eq. (19), we find

$$C_x(t) = \sum_{m=0}^{2q-1} \sum_{j=0}^{q-1} \sum_{k=1}^{2q} \frac{a_k^{m-j/q} \tilde{B}_{mj} A_k e^{a_k t}}{\Gamma(\frac{j}{q})} \gamma(\frac{j}{q}, a_k t) \quad (25)$$

where for $j = 0$, $\frac{\gamma(\frac{j}{q}, a_k t)}{\Gamma(\frac{j}{q})} = 1$, or using Eq. (22)

$$C_x(t) = \sum_{m=0}^{2q-1} \sum_{j=0}^{q-1} \sum_{k=1}^{2q} a_k^m \tilde{B}_{mj} A_k t^{\frac{j}{q}} E_{1, 1+\frac{j}{q}}(a_k t) \quad (26)$$

and for $j = 0$ $E_{1, 1+\frac{j}{q}}(a_k t) = e^{a_k t}$. Now we have a practical tool for finding an explicit analytical solution for the fractional damped harmonic oscillator. Since $\gamma(a, x)$ is tabulated in programs like Mathematica, the solution which is a finite sum of such incomplete gamma functions, can be plotted.

B. Method B

The task of finding $\hat{Q}(s)$ is sometimes difficult since as described in Sec. III A, generally one must solve a linear system of $2q(q-1)$ equations with $2q(q-1)$ variables. But for the special case of Eq. (7) we will provide a simple method for finding $\hat{Q}(s)$ and $\hat{P}(s)$. By writing

$$C_x(t) = \frac{s + \gamma s^{\frac{p}{q}-1}}{(s^2 + \omega^2)^q + (-1)^{q-1} \gamma^q s^p} \left(\frac{(s^2 + \omega^2)^q + (-1)^{q-1} \gamma^q s^p}{s^2 + \gamma s^{\frac{p}{q}} + \omega^2} \right) \quad (27)$$

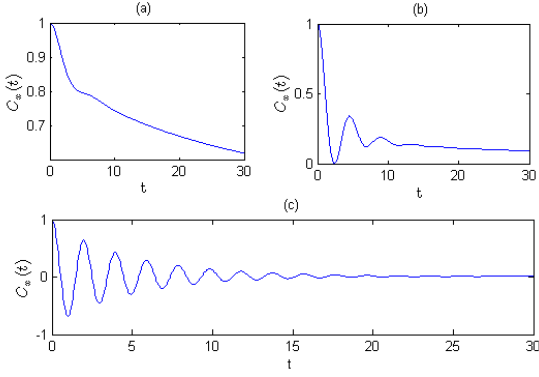


FIG. 1: The short time behavior for $C_x(t)$ with $\alpha = \frac{1}{2}$ and $\gamma = 1$, versus t . Three types of solutions are presented **(a)** $\omega = 0.3$ and the function decays monotonically. **(b)** $\omega = \omega_z \approx 1.053$ the transition between motion with and without zero crossing, $C_x(t) = 0$ at a single point in time, and $C_x(t)$ does not cross the zero line. **(c)** $\omega = 3$ the under-damped regime. Note that for large t , $C_x(t) > 0$.

we can write

$$\hat{Q}(s) = \frac{(s^2 + w^2)^q + (-1)^{q-1}\gamma^q s^p}{s^2 + \gamma s^{\frac{p}{q}} + w^2} \quad (28)$$

and

$$\hat{P}(s) = (s^2 + \gamma s^{\frac{p}{q}} + w^2) \hat{Q}(s) = (s^2 + w^2)^q + (-1)^{q-1}\gamma^q s^p. \quad (29)$$

One easily sees that indeed $\hat{P}(s)$ is a polynomial in s . As for $\hat{Q}(s)$, it is found by standard method of division of two polynomials in $s^{1/q}$ and it is also a polynomial in $s^{1/q}$. Since the fraction of two polynomials

$$\frac{(y^{2q} + w^2)^q + (-1)^{q-1}\gamma^q y^{qp}}{y^{2q} + \gamma y^p + w^2} \quad (30)$$

is also a polynomial in y , we see that any solution of $y^{2q} + \gamma y^p + w^2 = 0$ is also a solution of $(y^{2q} + w^2)^q + (-1)^{q-1}\gamma^q y^{qp} = 0$, and by performing a substitute $y = s^{\frac{1}{q}}$ into Eq. (30) we find that indeed $\hat{Q}(s)$ is a polynomial in $s^{1/q}$. After finding $\hat{Q}(s)$ and $\hat{P}(s)$ explicitly we can return to method A and use Eqs. (15-26) in order to write down the final solution.

IV. EXAMPLES $\alpha = \frac{1}{2}$ AND $\alpha = \frac{3}{4}$.

A. $\alpha = \frac{1}{2}$

The $\alpha = \frac{1}{2}$ case was recently measured in experiment [10], and so it will be our first illustration for the method. From Eq. (7) we get

$$\hat{C}_x(s) = \frac{s + \gamma s^{-\frac{1}{2}}}{s^2 + \gamma s^{\frac{1}{2}} + w^2}. \quad (31)$$

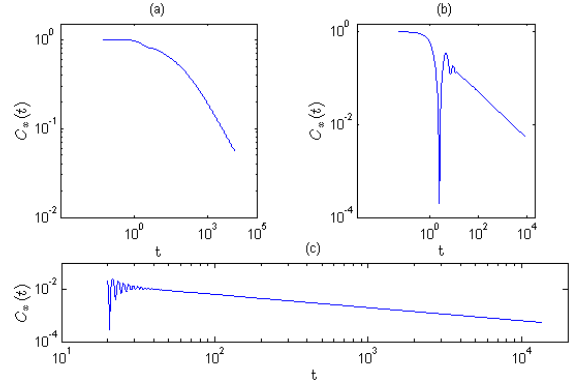


FIG. 2: The long time behavior for $C_x(t)$ with $\alpha = \frac{1}{2}$ and $\gamma = 1$, versus t , on a loglog scale. Three types of solutions are presented **(a)** $\omega = 0.3$ over-damped limit, monotonic decay. **(b)** $\omega = \omega_z \approx 1.053$ the transition between motion with and without zero crossing. **(c)** $\omega = 3$ the under damped regime, for large t , $C_x(t) > 0$ (for short time $C_x(t) < 0$, not shown). Notice that non-monotonic decay in the under-damped case is observed only for short times, while for long-times $C_x(t) \propto t^{-\frac{1}{2}}$.

The first step is to find the complementary polynomial of the denominator on the right hand-side of Eq. (31). Using method A of the previous section one can easily see that the coefficients of the complementary polynomial $\hat{Q}(s)$ are

$$B_4 = 1, B_3 = B_2 = 0, B_1 = -\gamma, B_0 = w^2$$

and

$$\hat{Q}(s) = s^2 - \gamma s^{\frac{1}{2}} + w^2 \quad (32)$$

since

$$\hat{P}(s) = \hat{Q}(s) (s^2 + \gamma s^{\frac{1}{2}} + w^2) = s^4 + 2w^2 s^2 - \gamma^2 s + w^4. \quad (33)$$

We rewrite Eq. (31) as

$$\hat{C}_x(s) = \frac{s^3 + s w^2 - \gamma^2 + \gamma w^2 s^{-\frac{1}{2}}}{(s^2 + w^2)^2 - \gamma^2 s} \quad (34)$$

and notice that the degree of the denominator of Eq. (34), i.e. $\hat{P}(s)$, is 4. Its zeros are easily found using Ferrari formula [32], and we call them a_k , $k = 1, \dots, 4$. The coefficients of the partial fraction expansion A_k are given by Eq. (16)

$$A_k = \frac{1}{4a_k(a_k^2 + w^2) - \gamma^2}. \quad (35)$$

The partial fraction expansion is found using Eq. (19)

$$\hat{C}_x(s) = \sum_{m=0}^3 \sum_{j=0}^1 \sum_{k=1}^4 \frac{a_k^m \tilde{B}_{mj} A_k}{s - a_k} s^{-\frac{j}{q}} \quad (36)$$

and the \tilde{B}_{mj} in Eq. (18) are found using the numerator of Eq. (34)

$$\tilde{B}_{30} = 1, \tilde{B}_{10} = \omega^2, \tilde{B}_{00} = -\gamma^2, \tilde{B}_{01} = \gamma\omega^2 \quad (37)$$

and other elements of the matrix \tilde{B}_{mj} are equal 0. Using Eq. (25) the solution is

$$C_x(t) = \sum_{k=1}^4 \left((-\gamma^2 + \omega^2 a_k + a_k^3) A_k e^{a_k t} + \gamma\omega^2 A_k t^{\frac{1}{2}} E_{1, \frac{3}{2}}(a_k t) \right). \quad (38)$$

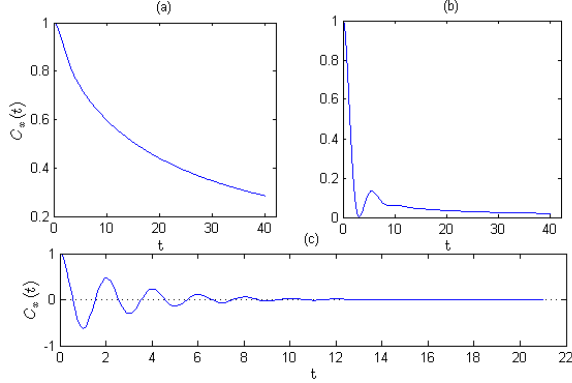


FIG. 3: The short time behavior for $C_x(t)$ with $\alpha = \frac{3}{4}$ and $\gamma = 1$, versus t . Three types of solutions are presented (a) $\omega = 0.3$ and the function decays monotonically. (b) $\omega = \omega_z \approx 0.965$ the transition between motion with and without zero crossing, $C_x(t)$ does not cross the zero line. (c) $\omega = 3$ the under-damped regime.

By using a series expansion and some algebra we find a_k and Eq. (38) has the following asymptotic behavior

$$C_x(t) = \left\{ \begin{array}{ll} 1 - \frac{1}{2}\omega^2 t^2 + \frac{\gamma\omega^2}{\Gamma(\frac{9}{2})} t^{\frac{7}{2}} & t \rightarrow 0 \\ \frac{\gamma}{\omega^2 \Gamma(\frac{1}{2})} t^{-\frac{1}{2}} - \left(\frac{\gamma}{\omega^2}\right)^3 \frac{t^{-\frac{3}{2}}}{2\Gamma(\frac{1}{2})} & t \rightarrow \infty \end{array} \right\}, \quad (39)$$

We see that $C_x(t)$ for long times decays as a power-law, which is the signature of slow relaxation and anomalous diffusion. The same asymptotic results are found by applying Tauberian Theorems [34] to Eq. (31).

The asymptotic expansions Eq. (39) provides the behavior for long (and short) times, but the intermediate behavior is not obvious. Using the exact solution Eq. (38) we plot $C_x(t)$ for various values of γ and ω in Fig. 1 and Fig. 2. Three types of behaviors exist (i) Monotonic decay of the solution - Fig. 1(a). (ii) Non-monotonic decay in the non-negative half of the plane, $C_x(t) \geq 0$, - Fig. 1(b). (iii) Oscillations of the solution, where $C_x(t)$ also takes negative values - Fig. 1(c). These are typical behaviors of the solution which we found also in other parameter set (not shown). From Fig. 1 we identify $\omega = \omega_z = 1.053$ as a fractional critical point, in the sense that if $\omega > \omega_z$ we have zero crossings for $C_x(t)$. For $\alpha = \frac{1}{2}$ there exist also another fractional critical point $\omega_m = 0.426$ where for $\omega < \omega_m$ $C_x(t)$ is monotonically decaying - Fig. 1(a). We will soon discuss ω_z and ω_m more generally.

B. $\alpha = \frac{3}{4}$

In this subsection we demonstrate the solution for $\alpha = \frac{3}{4}$ using method B. Our goal is to invert Eq. (7).

$$\hat{C}_x(s) = \frac{s + \gamma s^{\frac{3}{4}-1}}{s^2 + \gamma s^{\frac{3}{4}} + \omega^2}. \quad (40)$$

We find the complementary polynomial $\hat{Q}(s)$ and $\hat{P}(s)$, using Eq. (28) and Eq. (29)

$$\hat{P}(s) = (s^2 + \omega^2)^4 - \gamma^4 s^3 \quad (41)$$

and

$$\hat{Q}(s) = \frac{(s^2 + \omega^2)^4 - \gamma^4 s^3}{s^2 + \gamma s^{\frac{3}{4}} + \omega^2} = s^6 - \gamma s^{\frac{19}{4}} + 3\omega^2 s^4 + \gamma^2 s^{\frac{7}{2}} - 2\omega^2 \gamma s^{\frac{11}{4}} - \gamma^3 s^{\frac{9}{4}} + 3\omega^4 s^2 + \omega^2 \gamma^2 s^{\frac{3}{2}} - \gamma\omega^4 s^{\frac{3}{4}} + \omega^6 \quad (42)$$

and so we can write Eq. (40) as

$$\hat{C}_x(s) = \frac{s^7 + 3\omega^2 s^5 + \gamma\omega^2 s^{\frac{15}{4}} + 3\omega^4 s^3 - \gamma^2 \omega^2 s^{\frac{5}{2}} - \gamma^4 s^2 + 2\gamma\omega^4 s^{\frac{7}{4}} + \gamma^3 \omega^2 s^{\frac{5}{4}} + \omega^6 s - \gamma^2 \omega^4 s^{\frac{1}{2}} + \gamma\omega^6 s^{-\frac{1}{4}}}{(s^2 + \omega^2)^4 - \gamma^4 s^3}. \quad (43)$$

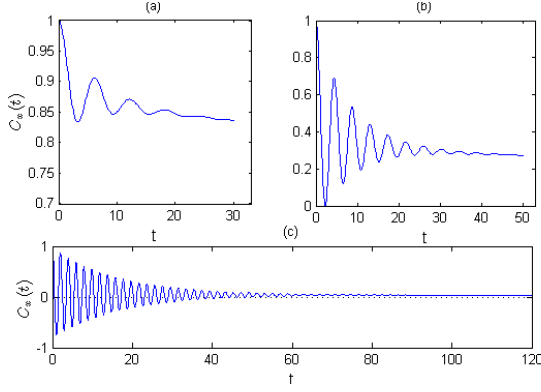


FIG. 4: The short time behavior for $C_x(t)$ with $\alpha = \frac{1}{5}$ and $\gamma = 1$, versus t . The three types of solutions are presented **(a)** $\omega = 0.3$ and the function oscillates above zero. **(b)** $\omega = \omega_z = 1.035$ the transition between motion with and without zero crossing, $C_x(t)$ does not cross the zero line. **(c)** $\omega = 3$ oscillations with zero crossing for short times, for long times $C_x(t) > 0$ and no oscillations are observed. For this case solution with monotonic decay are not found since $\alpha = \frac{1}{5} < \alpha_c = 0.402$.

The degree of the denominator of Eq. (43) is 8, so the zeros of the polynomial $\hat{P}(s)$ could only be found numerically using a program like Mathematica. As in the previous subsection we call the zeros of $\hat{P}(s)$ a_k , $k = 1, \dots, 8$ and the coefficients of partial fraction expansion A_k are found using Eq. (16)

$$A_k = \frac{1}{8a_k(a_k^2 + \omega^2)^3 - 3\gamma^4 a_k^2}. \quad (44)$$

Writing the partial fraction expansion using Eq. (19)

$$\hat{C}_x(s) = \sum_{m=0}^7 \sum_{j=0}^3 \sum_{k=1}^8 \frac{a_k^m \tilde{B}_{mj} A_k}{s - a_k} s^{-\frac{j}{q}}, \quad (45)$$

where the coefficients \tilde{B}_{mj} are found using the numerator of Eq. (43)

$$\tilde{B} = \begin{pmatrix} 0 & \gamma\omega^6 & 0 & 0 \\ \omega^6 & 0 & -\gamma^2\omega^4 & 0 \\ -\gamma^4 & 2\gamma\omega^4 & 0 & \gamma^3\omega^2 \\ 3\omega^4 & 0 & -\gamma^2\omega^2 & 0 \\ 0 & \gamma\omega^2 & 0 & 0 \\ 3\omega^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \quad (46)$$

Using Eq. (25) the solution is

$$C_x(t) = \sum_{k=1}^8 A_k e^{a_k t} (a_k \omega^6 - a_k^2 \gamma^4 + 3a_k^3 \omega^4 + 3a_k^5 \omega^2 + a_k^7) + A_k e^{a_k t} \left(\frac{\gamma \gamma(\frac{1}{4}, a_k t)}{\Gamma(\frac{1}{4})} (\omega^6 a_k^{-\frac{1}{4}} + 2\omega^4 a_k^{\frac{7}{4}} + \omega^2 a_k^{\frac{15}{4}}) + \frac{\gamma^2 \gamma(\frac{1}{2}, a_k t)}{\Gamma(\frac{1}{2})} (\omega^4 a_k^{\frac{1}{2}} + \omega^2 a_k^{\frac{5}{2}}) + \frac{\omega^2 a_k^{\frac{5}{4}} \gamma^3 \gamma(\frac{3}{4}, a_k t)}{\Gamma(\frac{3}{4})} \right) \quad (47)$$

or using Eq. (26)

$$C_x(t) = \sum_{k=1}^8 A_k e^{a_k t} (a_k \omega^6 - a_k^2 \gamma^4 + 3a_k^3 \omega^4 + 3a_k^5 \omega^2 + a_k^7) + A_k \left(\gamma t^{\frac{1}{4}} E_{1, \frac{5}{4}}(a_k t) (\omega^6 + 2\omega^4 a_k^2 + \omega^2 a_k^4) + \gamma^2 t^{\frac{1}{2}} E_{1, \frac{3}{2}}(a_k t) (\omega^4 a_k + \omega^2 a_k^3) + \omega^2 a_k^2 t^{\frac{3}{4}} \gamma^3 E_{1, \frac{7}{4}}(a_k t) \right). \quad (48)$$

The behavior described by Eq. (48) is plotted at Fig. 3. The three typical types of behavior are shown, which are

similar to the behavior for the $\alpha = \frac{1}{2}$ case Fig. 1. The

values of critical points for $\alpha = \frac{3}{4}$ are $\omega_m \approx 0.707$ (and so for $\omega = 0.3$ we observe in Fig. 3(a) a monotonic decay) and $\omega_z \approx 0.965$, a case plotted in Fig. 3(b). Finally the case $\alpha = \frac{1}{5}$ is shown in Fig. 4. The difference between $\alpha = \frac{1}{5}$ and the former cases is that for $\alpha = \frac{1}{5}$ $\omega_m = 0$ and so we never observe a monotonic decay of the solution. More generally, this behavior is obtained for any $\alpha < \alpha_c \simeq 0.402$, as we shall soon show.

V. DEFINITION OF OVER AND UNDER DAMPED MOTION

As mentioned in the introduction, when dealing with the normal damped oscillator one gets two types of solutions - over-damped and under-damped, the transition between these two behaviors happens at some point ω_c called the critical point. For the over-damped motion $\langle x \rangle > 0$ for any t when $\langle x(t=0) \rangle > 0$, and there are no oscillations, and for the under-damped case $\langle x(t) \rangle$ oscillates and crosses the zero line. For the fractional oscillator, we notice that there are different types of behaviors. From Figs. (1,3,4) one notices that for short times there is an oscillating behavior either with and without zero-crossings or a monotonic decay type of behavior. So as in the regular damped oscillator we need to define the transition between these behaviors. We propose three definitions for the point of transition between over-damped and under-damped motions, these are based on the various definitions that exist for the regular damped oscillator and give the same result for $\alpha = 1$. The first option is to take the frequency ω_c for which two solutions of $\hat{P}(s) = 0$ coincide, i.e. an appearance of a pole of a second order for $\hat{C}_x(s)$, and the general solution of Sec. III must be modified, as explained in Appendix VIII. The second option is to take the minimal frequency ω_z at which the solution $C_x(t)$ crosses the zero line and the third is to take the minimal frequency ω_m at which $\frac{dC_x(t)}{dt}$ crosses the zero line (i.e. $C_x(t)$ is no longer a monotonically decaying function). For regular damped oscillator $\omega_c = \omega_z = \omega_m$, but in fractional case this is generally not true.

Another difference between the fractional oscillator and the regular one is the distinction between short and long time behavior when $0 < \alpha < 1$. The asymptotic behavior for general α is obtained using general properties of polynomial solutions [18] with Eq. (23) and Eq. (24) or using the Tauberian theorem [34]

$$C_x(t) \approx 1 - \frac{1}{2}\omega^2 t^2 + \frac{\omega^2 \gamma}{\Gamma(5-\alpha)} t^{4-\alpha} \quad t \rightarrow 0 \quad (49)$$

$$C_x(t) \approx \frac{\gamma}{\omega^2 \Gamma(1-\alpha)} t^{-\alpha} \quad t \rightarrow \infty \quad (50)$$

where the large t expression was obtained in [10, 12, 15]. The applicability of Eq. (50) is possible only under two conditions, the first one is obvious from Eq. (50) and is

$$\left(\frac{\omega^2}{\gamma}\right)^{1/\alpha} t \gg 1, \quad (51)$$

while the second is

$$|a|t \gg 1, \quad (52)$$

where $|a|$ is an absolute value of a zero of the enumerator in Eq. (7) ($s^2 + \gamma s^\alpha + \omega^2$), it also could be found from $\hat{P}(s) = 0$. Mathematically the second condition is the magnitude of the radius of convergence for the power series expansion of $\hat{C}_x(s)$ near $s = 0$ (Tauberian Theorem), and is given by the non-fictitious poles of $\hat{C}_x(s)$. From Eq. (50), $C_x(t) > 0$ and for large t decays as a power-law (see Fig. 2).

From a more physical point of view, the FLE formalism was used [10] to describe the fluctuation of the distance between a fluorescein-tyrosine pair within a single protein on a time scales of 1 msec up to 10^2 sec, and a power-law decay was observed (lately a theoretical model of Fractons was proposed in order to explain such phenomena [40]). Recent molecular dynamics simulations [41] studied the fluctuations of a donor-acceptor distance for a single protein, for short time scales 10^{-9} sec and oscillations of the autocorrelation function were observed. The scenario of oscillations for short-times and a power-law decay for long-times set's well with the description by our solutions of the FLE.

A. Critical point ω_c

For the case $\alpha = \frac{1}{2}$ we have four solutions for $\hat{P}(s) = 0$ (a_k $k = 1, \dots, 4$) which are plotted in Fig. 5. At one point two solutions coincide, we call this point the critical point at which $\omega = \omega_c$. For $\omega = \omega_c$, the solution for $C_x(t)$ Eq. (38) must be modified because our general method derived Sec III is not valid. For $\alpha = \frac{1}{2}$ and $\omega = \omega_c = \frac{1}{2^{2/3}} \sqrt{(3/2)(1/2)^{1/3} \gamma^{2/3}}$ (see Eq. 59) Eq. (38) is (see Appendix, Eq. (95))

$$C_x(t) = \sum_{m=0}^3 \sum_{j=0}^1 \tilde{B}_{mj} t^{\frac{j}{2}} \left[\sum_{k=1}^3 a_k^m A_k E_{1,1+\frac{j}{2}}(a_k t) + a_3^m \tilde{A} \left((t + m a_3^{-1}) E_{1,1+\frac{j}{2}}(a_3 t) - \frac{j}{2} t E_{1,2+\frac{j}{2}}(a_3 t) \right) \right] \quad (53)$$

where $a_1 \neq a_2 \neq a_3$ and $a_4 = a_3$ are zeros of $\hat{P}(s)$ given by Eq. (33), \hat{B}_{mj} are found using Eq. (37), A_1 and A_2 by Eq. (35), $A_3 = -(A_1 + A_2)$ by Eq. (87) and \hat{A} is defined in Eq. (88).

We emphasize that the critical point ω_c does not always exist. In Fig. 6 we plot the 10 solutions of $\hat{P}(s) = 0$ for $\alpha = \frac{2}{5}$, and demonstrate that no two solutions coincide. We will soon show that for any odd q and even p the critical point ω_c does not exist.

Mathematically at critical points one of the coefficients of partial fractions expansion, A_k in Eq. (16), diverges. This happens because when two a_k coincide, $\hat{P}(s)$ can be written as $(s - a_k)^2 \hat{G}(s)$, where $\hat{G}(s)$ is some polynomial in s and $G(a_k) \neq 0$, and

$$A_k = \frac{1}{\left(2(s - a_k)\hat{G}(s) + (s - a_k)^2 \frac{d\hat{G}(s)}{ds}\right) \Big|_{s=a_k}} \quad (54)$$

diverges. And so in order to find such critical point two conditions must be satisfied

$$\hat{P}(s) = 0, \quad \frac{d\hat{P}(s)}{ds} = 0. \quad (55)$$

Using Eq. (29) with $\alpha = \frac{p}{q}$ and $\omega = \omega_c$,

$$(s^2 + \omega_c^2)^q + (-1)^{q-1} \gamma^q s^p = 0 \quad (56)$$

$$2qs(s^2 + \omega_c^2)^{q-1} + p(-1)^{q-1} \gamma^q s^{p-1} = 0. \quad (57)$$

Solving these Eqs. we find

$$s = \pm \sqrt{\frac{\alpha}{2 - \alpha}} \omega_c \quad (58)$$

and ω_c must satisfy

$$\omega_c = \frac{1}{2^{\frac{1}{2-\alpha}}} \sqrt{(2 - \alpha) \alpha^{\frac{\alpha}{2-\alpha}} \gamma^{\frac{1}{2-\alpha}}}. \quad (59)$$

Eq. (58) and Eq. (59) are only valid for even q or even $(q + p)$ (recall $\alpha = \frac{p}{q}$ and $\frac{p}{q}$ is irreducible), since for any other case Eqs. (56,57) has no solutions. The + sign in Eq. (58) is for the case of even q and odd p and the - sign for odd q and p . To see this, insert the solution Eq. (58) in Eq. (56) and then we must have $(-1)^{q-1} < 0$ and hence q even, since $\frac{p}{q}$ is irreducible p is odd. Similarly for the - solution in Eq. (58). From this discussion it becomes clear why there are no critical frequency ω_c for $\alpha = \frac{2}{5}$ (Fig. 6).

B. Critical points ω_z and ω_m

We divide the phase space to three different regions. (I) $0 < \omega < \omega_m$ the region where $C_x(t)$ decays monotonically (II) $\omega_m < \omega < \omega_z$ the region of non-monotonic

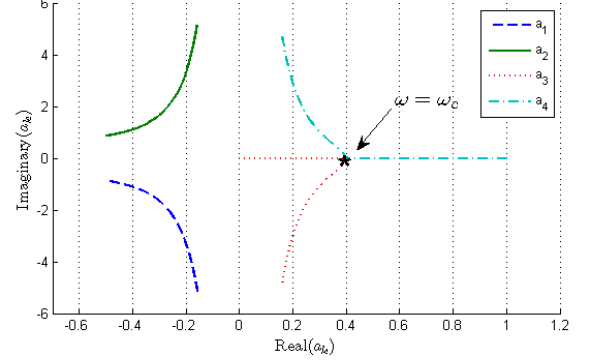


FIG. 5: The four solutions of $\hat{P}(a_k) = 0$ in the imaginary plane for $\alpha = \frac{1}{2}$, $\gamma = 1$ and $0 < w < 5$. At the critical point $\omega = \omega_c = 0.6873$ we have $a_3 = a_4$.

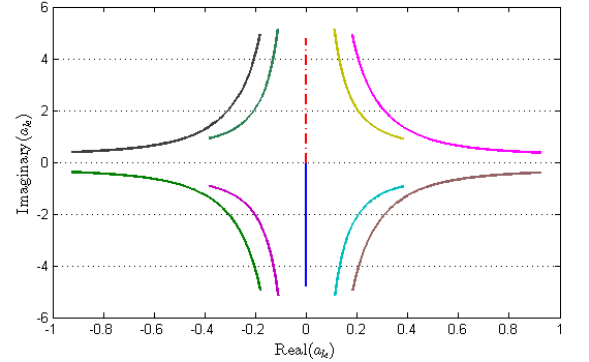


FIG. 6: The ten solutions of $\hat{P}(a_k) = 0$ for $\alpha = \frac{2}{5}$, and $\gamma = 1$, in the imaginary plane, for $0 < w < 5$. All ten solutions are different and do not coincide, namely for this case there does not exist a critical frequency ω_c since $q = 5$ is odd. The origin corresponds to $\omega = 0$.

decay while $C_x(t)$ always stays positive (III) $\omega_z < \omega$ the region of non-monotonic decay while part of the time $C_x(t)$ is negative. Similar to Eq. (59) we find from dimensional analysis

$$\omega_z = \kappa_z(\alpha) \gamma^{\frac{1}{2-\alpha}} \quad (60)$$

and

$$\omega_m = \kappa_m(\alpha) \gamma^{\frac{1}{2-\alpha}} \quad (61)$$

where $\kappa_z(\alpha)$ and $\kappa_m(\alpha)$ depend only on α . By investigating analytical solution Eq. (26) for various α and $\gamma = 1$ we obtain functions $\kappa_z(\alpha)$ and $\kappa_m(\alpha)$. The resulting phase diagram is presented in Fig. 7. One can readily see that ω_z , ω_m and ω_c defined by Eq. (59) all coincide only for the normal case $\alpha = 1$.

An interesting behavior is observed for $\kappa_m(\alpha)$, as can be seen in Fig. 7 a sort of phase transition occurs around $\alpha \approx 0.4$. We used the general method developed in

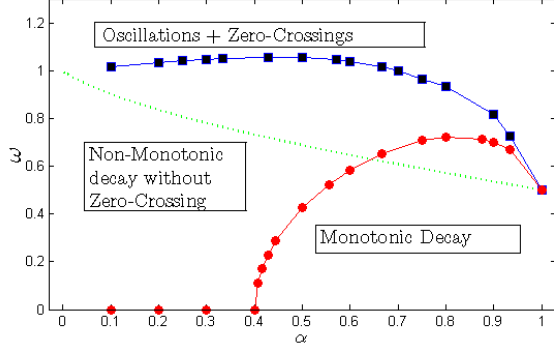


FIG. 7: The phase diagram of the fractional oscillator. Phase (a) monotonic decay of the correlation function $C_x(t)$, phase (b) non-monotonic decay without zero-crossing and (c) oscillations with zero crossings. The boundary between (b) and (c) is $\omega_z = \kappa_z(\alpha)$ (solid line + squares), the boundary between (a) and (b) is $\omega_m = \kappa_m(\alpha)$ (solid line + circles). For $\alpha < \alpha_c \simeq 0.402$, the phase of monotonic decay disappears, namely we do not find over damped behavior. The dotted curve is the critical line ω_c given by Eq. (59). All the curves are calculated for $\gamma = 1$. For $\alpha = 1$, $\omega_c = \omega_z = \omega_m = \gamma/2$.

Sec. III and explored the behavior of $\frac{1}{\omega^2} \frac{dC_x(t)}{dt}$, which in Laplace space is given by

$$\frac{1}{\omega^2} [s\hat{C}_x(s) - 1] = -\frac{1}{s^2 + \gamma s^\alpha + \omega^2}. \quad (62)$$

For $\alpha \leq 0.4$ we always observed zero crossings even if we decreased ω to 10^{-7} ($\gamma = 1$), while for $\alpha \geq 0.404$ there were no zero crossings beneath some finite $\omega > 0$, as is shown in Fig. 7. So we can conclude that there exist a critical α , $\alpha_c \approx 0.402 \pm 0.002$, where for $\alpha < \alpha_c$ $C_x(t)$ does not decay monotonically even if the frequency of the binding Harmonic field $\omega \rightarrow 0$. Note that the phase diagram Fig. 7 also exhibits some expected behaviors: as we increase ω we find a critical line above which the solutions are non-monotonic and exhibit zero crossing, a line represented by $\kappa_z(\alpha)$.

To find accurate values of α_c we also used a method based on Bernstein theorem [34]. According to the theorem if and only if $f(t)$ is positive then for any integer n

$$0 \leq (-1)^n \frac{d^n \tilde{f}(s)}{ds^n} \quad (s > 0), \quad (63)$$

where $\tilde{f}(s)$ is the Laplace pair of $f(t)$. As in the previous paragraph in order to check the monotonicity of $C_x(t)$, we will inspect $\frac{1}{\omega^2} \frac{dC_x(t)}{dt}$ for zero crossings, or by speaking in the language of Bernstein theorem by exploring $(-1)^n \frac{d^n \tilde{f}(s)}{ds^n}$, when $-\tilde{f}(s)$ is given by Eq. (62). Using the scaling relation of Eq. (61) we can set $\gamma = 1$, and so it is easily checked that for $n = 0$ and $n = 1$ $(-1)^n \frac{d^n \tilde{f}(s)}{ds^n} > 0$,

for any $0 < \alpha < 1$. But for $n = 2$

$$(-1)^n \frac{d^n \tilde{f}(s)}{ds^n} = \frac{1}{(s^2 + s^\alpha + \omega^2)^3} (6s^2 + (9\alpha - \alpha^2 - 2)s^\alpha + (\alpha^2 + \alpha)s^{2\alpha-2} - \omega^2(2 + \alpha(\alpha - 1))s^{\alpha-2}), \quad (64)$$

and in the limit $\omega \rightarrow 0$ one can easily show that for $\alpha \leq 0.071$ Eq.(64) has negative values. Actually for $\alpha = 0.01$ it is easily verified (by plotting Eq.(64)) that for any $\omega > 0$ Eq. (64) would have negative values. So for $n = 2$ we have a upper bound for α , $\alpha^{(2)} \approx 0.071$, where for any $\alpha < \alpha^{(2)}$ $\frac{dC_x(t)}{dt}$ crosses the zero line and the relaxation is non-monotonic. If one wishes to increase the accuracy for such an upper bound one should inspect the behavior of $(-1)^n \frac{d^n \tilde{f}(s)}{ds^n}$ for higher values of n , for any n we define such a bound as $\alpha^{(n)}$. Using Mathematica we can proceed to higher values of n and find the upper bound $\alpha^{(n)}$ for different ω . In Fig. 8 we plotted the upper bounds as a function of n for various ω , we see that as n grows the upper bound $\alpha^{(n)}$ converges to some value < 1 . The results achieved by this method converge to values very close to the values displayed in Fig. 7, for example for $\omega = 0$ and $n = 150$ $\alpha^{(150)} = 0.394$, compared with $\alpha_c \approx 0.402$, obtained by inspection of exact solution.

A physical explanation for this interesting result is based on the cage effect. For small α the friction force induced by the medium is not just slowing down the particle but also causing the particle a rattling motion. To see this consider the FLE Eq. (11) in the limit $\alpha \rightarrow 0$

$$m\ddot{x} + m\gamma(x - x_0) + m\omega^2 x \approx \xi(t) \quad (65)$$

where x_0 is the initial condition. Eq.(65) describes harmonic motion and the friction γ in this $\alpha \rightarrow 0$ limit yields an elastic harmonic force. In this sense the medium is binding the particle preventing diffusion but forcing oscillations. In the opposite limit of $\alpha \rightarrow 1$

$$m\ddot{x} + m\gamma\dot{x} + m\omega^2 x \approx \xi(t), \quad (66)$$

the usual damped oscillator with noise is found. So from Eq. (65) an oscillating behavior is expected, even when $\omega \rightarrow 0$ which can be explained by the rattling motion of a particle in the cage formed by the surrounding particles. This behavior manifests in the non-monotonic oscillating solution we have found for small α . Our findings that α_c marks a non smooth transition between normal friction $\alpha \rightarrow 1$ and elastic friction $\alpha \rightarrow 0$ is certainly a surprising result.

VI. RESPONSE TO AN EXTERNAL FIELD

In this section we will explore the response of the FLE to an external time dependent force. The response of a system to an oscillating time dependent field naturally leads to the phenomena of resonances, when the frequency of the external field matches a natural frequency

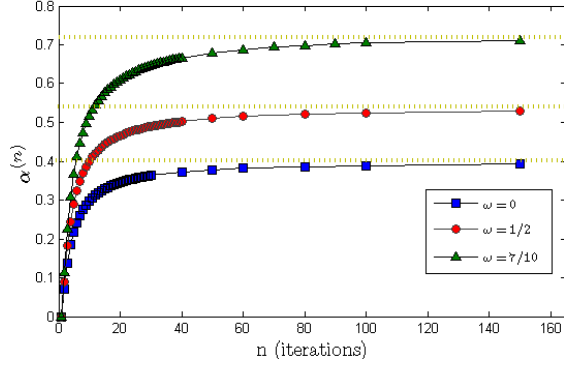


FIG. 8: Using the Bernstein Method we find the upper bound $\alpha^{(n)}$ as a function of n , the number of the derivatives. We consider three different ω ($\omega = 0$, $\omega = \frac{1}{2}$ and $\omega = \frac{7}{10}$). The dashed lines present the values found numerically in Fig. 7. As n grows a convergence of the bound is achieved as is seen for $\omega = 0$ as $\alpha^{(n)}$ converges to $\alpha_c \approx 0.402 \pm 0.002$.

of the system. The response of sub-diffusing systems to such time dependent field was the subject of intensive research [35, 36, 37]. In particular fractional approach to sub-diffusion naturally leads to anomalous response functions commonly found in many systems e.g. the Cole-Cole relaxation [24, 38, 39]. In the next three subsections we will explore the response of FLE with and without the Harmonic potential and also investigate the behavior of the imaginary part of the complex susceptibility, i.e. the dielectric loss, for these cases.

Our starting point is Eq. (2) with $F(x, t) = F_0 \cos(\Omega t) - m\omega^2 x$, performing an average we obtain

$$\langle \ddot{x} \rangle + \gamma \frac{d^\alpha \langle x \rangle}{dt^\alpha} + \omega^2 \langle x \rangle = \frac{F_0}{m} \cos(\Omega t). \quad (67)$$

The solution in the long time regime is

$$\langle x(t) \rangle \sim \frac{F_0}{m} \int_0^t \cos(\Omega(t-\tau)) h(\tau) d\tau, \quad (68)$$

where $h(t)$ is soon defined. Eq. (68) could be written as

$$\langle x(t) \rangle = R(\Omega) \cos(\Omega t + \theta(\Omega)) \quad t \rightarrow \infty. \quad (69)$$

The response $R(\Omega)$ and the phase shift $\theta(\Omega)$ are obtained by the means of complex susceptibility

$$\chi(\Omega) = \chi'(\Omega) + i\chi''(\Omega) = \hat{h}(-i\Omega) \quad (70)$$

where $\chi'(\Omega)$ and $\chi''(\Omega)$ are the complex and the imaginary parts of the susceptibility, respectively, and $\hat{h}(-i\Omega) = \int_0^\infty e^{i\Omega t} h(t) dt$ [16]. For the response

$$R(\Omega) = |\chi(\Omega)| \quad (71)$$

and

$$\theta(\Omega) = \arctan\left(-\frac{\chi''(\Omega)}{\chi'(\Omega)}\right), \quad (72)$$

for the phase shift.

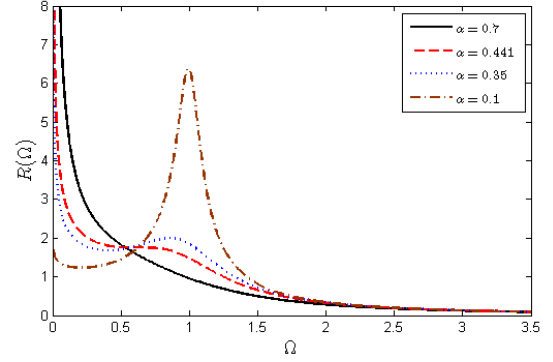


FIG. 9: The response of FLE to an oscillating field for a free particle and different α 's. For $\alpha < \alpha_R$ a resonance is observed. All the curves are plotted with $\gamma = 1$.

A. Unbounded Particle

For the unbounded particle we set $\omega = 0$ in Eq. (67) and using Eq. (4) we obtain for $\chi(\Omega)$

$$\chi(\Omega) = \hat{h}(-i\Omega) = \frac{1}{\gamma(-i\Omega)^\alpha - \Omega^2}. \quad (73)$$

By the use of Eq. (71) we can explicitly obtain now the behavior of $R(\Omega)$ for any $0 < \alpha < 1$. For the normal diffusion case when $\alpha = 1$, the response $R(\Omega)$ is a decaying function of Ω and no resonance is observed, but the picture is quite different for $0 < \alpha < 1$. In this sub-diffusive part the response $R(\Omega)$ is not always a monotonically decaying function and could obtain a maximum, i.e. a resonance, even for such a free motion. As is seen in Fig. 9 for small enough α , $R(\Omega)$ has a resonance, we will show that the existence of the resonance does not depend on any other parameter but α .

Writing down the response $R(\Omega)$ explicitly

$$R(\Omega) = \frac{1}{\sqrt{\Omega^4 + \gamma^2 \Omega^{2\alpha} - 2\gamma \Omega^{2+\alpha} \cos\left(\frac{\pi\alpha}{2}\right)}}, \quad (74)$$

We are looking for the solutions of $dR(\Omega)/d\Omega = 0$, and hence

$$4\Omega_R^3 + 2\alpha\gamma^2\Omega_R^{2\alpha-1} - 2(\alpha+2)\gamma\Omega_R^{\alpha+1} \cos\left(\frac{\pi\alpha}{2}\right) = 0, \quad (75)$$

so the solution is

$$\begin{aligned} \frac{\gamma}{\Omega_R^{2-\alpha}} \\ = \frac{1}{2\alpha} \left((\alpha+2) \cos\left(\frac{\pi\alpha}{2}\right) \pm \sqrt{(\alpha+2)^2 \cos^2\left(\frac{\pi\alpha}{2}\right) - 8\alpha} \right), \end{aligned} \quad (76)$$

and Ω_R is the frequency for which the resonance is found. When the discriminant on the right hand of Eq. (76) is greater than zero there will be a resonance. The discriminant has no dependence on γ , and is always positive for

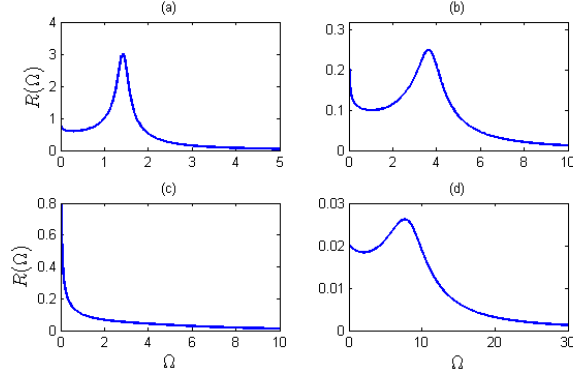


FIG. 10: The response of FLE to an oscillating field for a harmonically bounded particle and different α s. (a) $\alpha = 0.2$, $\omega = 1$ and $\gamma = 1$. (b) $\alpha = 0.2$, $\omega = 1$ and $\gamma = 10$. (c) $\alpha = 0.7$, $\omega = 1$ and $\gamma = 10$. (d) $\alpha = 0.7$, $\omega = 7$ and $\gamma = 10$.

$\alpha \leq \alpha_R = 0.441021\dots$ which satisfies the following relation

$$(\alpha_R + 2)^2 \cos^2\left(\frac{\pi\alpha_R}{2}\right) - 8\alpha_R = 0 \quad (77)$$

So no resonance is found for $\alpha > \alpha_R$ and for $\alpha < \alpha_R$ there is always exist an $\Omega_R > 0$ for which the response will exhibit a resonance. This result of a resonance for a free particle FLE is highly unexpected but sets well with our description of the friction force for small α 's as an elastic force due to the cage effect.

B. Harmonically Bounded Particle

Now we treat the response function of the FLE with a harmonic field, i.e. of a fractional oscillator. Starting with Eq. (67) we set the initial conditions $x_0 = v_0 = 0$ and in the long time limit $t \rightarrow \infty$ we obtain again $\langle x \rangle = R(\Omega) \cos(\Omega t + \theta(\Omega))$ where $R(\Omega)$ and $\theta(\Omega)$ are defined by Eqs. (71,72). The complex susceptibility for such case is obtained

$$\chi(\Omega) = \frac{1}{(\omega^2 - \Omega^2) + \gamma(-i\Omega)^\alpha}. \quad (78)$$

Eq. (78) was already obtained [42, 43, 44] for the fractional Klein-Kramers equation [21] in the high damping limit and is called a generalized Rocard equation [45, 46]. For $\alpha = 1$ we find the complex susceptibility of a normal damped oscillator.

We are interested in the resonance points for the response to the applied field, i.e. points of maximum of $R(\Omega)$, which generally depend on Ω , γ and ω (see Fig. 10). For the normal oscillator there is a resonance if the condition $\omega \geq \frac{1}{\sqrt{2}}\gamma$ is satisfied. If this condition is not satisfied, $R(\Omega)$ is a monotonically decreasing function of Ω . From

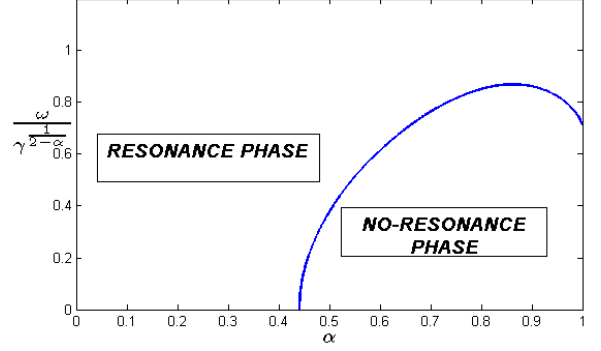


FIG. 11: Phase diagram of the response of the system with a harmonic field to a oscillating time dependent force field. Two simple behaviors are found either a resonance exists (“Resonance Phase”) or not (“No-Resonance Phase”). For $\alpha < \alpha_R = 0.441\dots$ a resonance exists for any binding harmonic field and any friction γ .

Eqs. (71,78)

$$R(\Omega) = \frac{1}{\sqrt{(\omega^2 - \Omega^2)^2 + \gamma^2 \Omega^{2\alpha} + 2(\omega^2 - \Omega^2)\gamma \Omega^\alpha \cos\left(\frac{\pi\alpha}{2}\right)}}, \quad (79)$$

and using $dR(\Omega)/d\Omega = 0$, we find

$$2\alpha\left(\frac{\gamma}{\Omega_R^{2-\alpha}}\right)^2 + [2\alpha\left(\frac{\omega^2}{\Omega_R^2} - 1\right) - 4]\cos\left(\frac{\pi\alpha}{2}\right)\frac{\gamma}{\Omega_R^{2-\alpha}} - 4\left(\frac{\omega^2}{\Omega_R^2} - 1\right) = 0. \quad (80)$$

The exploration of Eq. (80) (see Appendix IX) leads to two findings which are presented in Fig. 11. The first one is the existence of the same critical α_R given by Eq. (77) for the response of FLE with a harmonic potential, i.e. for any $\alpha < \alpha_R$ there always exist a specific Ω_R (which depends on γ and ω) for which the system is in resonance. The second finding is that above α_R we have a well defined boundary between the phase for which the resonance exist - a “Resonance Phase”, and a phase where are no resonance - “No-Resonance Phase” (see Fig. 11). For $\alpha < \alpha_R$ “No-Resonance-Phase” does not exists. The boundary is given by the following relation

$$\frac{\omega}{\gamma^{\frac{1}{2-\alpha}}} = g(\alpha), \quad (81)$$

and the phase diagram is presented in Fig. 11. The function $g(\alpha)$ is found analytically (see Appendix IX)

$$g(\alpha) = c(\alpha)^{-\frac{1}{2-\alpha}} \sqrt{\frac{c(\alpha)^2 - c(\alpha) \cos\left(\frac{\pi\alpha}{2}\right) \left(1 + \frac{2}{\alpha}\right) + \frac{2}{\alpha}}{\frac{2}{\alpha} - c(\alpha) \cos\left(\frac{\pi\alpha}{2}\right)}} \quad (82)$$

where $c(\alpha)$ is given by Eq. (108). As is seen from Fig. 11 for $\omega/\gamma^{\frac{1}{2-\alpha}} > g(\alpha)$ a “Resonance Phase” is obtained and “No-Resonance Phase” for $\omega/\gamma^{\frac{1}{2-\alpha}} < g(\alpha)$. In the limit

of $\alpha \rightarrow 1$ the boundary goes to the expected value for damped oscillator $1/\sqrt{2}$. Near the critical point α_R the $g(\alpha)$ drops to zero as a power-law with exponent $1/2$

$$g(\alpha) \propto (\alpha - \alpha_R)^{\frac{1}{2}} \quad \alpha \rightarrow \alpha_R. \quad (83)$$

The existence of the same critical α_R for a free and a harmonically bounded particle is easily understood from the phase diagram in Fig. 11. Choosing the straight line $\omega = 0$, which represent the free particle, and going along this line when starting at the “Resonance Phase” we will cross to the “No-Resonance Phase” exactly for $\alpha = \alpha_R$. Generally speaking the same critical α_R will be obtained for any kind of external force because it is determined by the internal properties of the surrounding medium represented by the friction part in FLE. The phase diagram

in Fig. 11 has much in common with the phase diagram in Fig. 7, which is quite reasonable because of the strong connection between non-monotonic decay of the correlation $C_x(t)$ and existence of a resonance. A presence of a resonance for small enough α emphasize the previously obtained result of non-existence of the over-damped limit for such α . Those properties are due to the same cage effect that we already discussed.

C. Complex Susceptibility

Eq. (78) for the complex susceptibility can be written in the following form

$$\chi(\Omega) = \chi'(\Omega) + i\chi''(\Omega) = \frac{(\omega^2 - \Omega^2) + \gamma\Omega^\alpha \cos\left(\frac{\pi\alpha}{2}\right)}{(\omega^2 - \Omega^2)^2 + \gamma^2\Omega^{2\alpha} + 2\gamma(\omega^2 - \Omega^2)\Omega^\alpha \cos\left(\frac{\pi\alpha}{2}\right)} + i \frac{\gamma\Omega^\alpha \sin\left(\frac{\pi\alpha}{2}\right)}{(\omega^2 - \Omega^2)^2 + \gamma^2\Omega^{2\alpha} + 2\gamma(\omega^2 - \Omega^2)\Omega^\alpha \cos\left(\frac{\pi\alpha}{2}\right)}, \quad (84)$$

the real and the imaginary parts of the susceptibility are experimentally measured quantities for many systems and so it is interesting to explore their behavior for the FLE. In this subsection we will explore the behavior of the imaginary part $\chi''(\Omega)$ which is also called “the loss”. From Fig. 12 we observe interesting behavior of $\chi''(\Omega)$ for different α ’s, not only one peak is present as is expected for the normal oscillator, but we observe a double peak phenomena for some α ’s. The double peak phenomena of “the loss” for super-cooled liquids and protein solutions is a well known phenomena [47, 48, 49] and usually treated by the means of mode-coupling theory [48]. We define two phases for the behavior of $\chi''(\Omega)$ and in the following find the phase diagram for $\chi''(\Omega)$. The first phase is the phase where $\chi''(\Omega)$ has only one peak - “One-Peak” phase and a “Double-Peak” phase, where a double peak phenomena is observed.

In Appendix X we explored $d\chi''(\Omega)/d\Omega$ and searched for the boundaries between the “One-Peak” and “Double-Peak” phases. The result is presented in Fig. 13. The boundaries between the phases are given by analytical functions $\tilde{g}_1(\alpha)$ and $\tilde{g}_2(\alpha)$ which are only α dependent. Two critical α are found for such a phase diagram, the first one $\alpha_{\chi_1} = 0.527\dots$ for which the boundary $\tilde{g}_1(\alpha)$ drops to zero, and a second one $\alpha_{\chi_2} = 0.707\dots$ for which $\tilde{g}_1(\alpha)$ and $\tilde{g}_2(\alpha)$ coincide (see Fig. 12). We also must note that near α_{χ_1} $\tilde{g}_1(\alpha)$ behaves like $\tilde{g}_1(\alpha) \propto (\alpha - \alpha_{\chi_1})^{1/2}$, a behavior which was also observed for $g(\alpha)$ and $\kappa_m(\alpha)$ near the corresponding critical points.

The double-peak phenomena in our case is explained in the same sense as was explained the existence of resonance for small enough α , and the disappearance of the

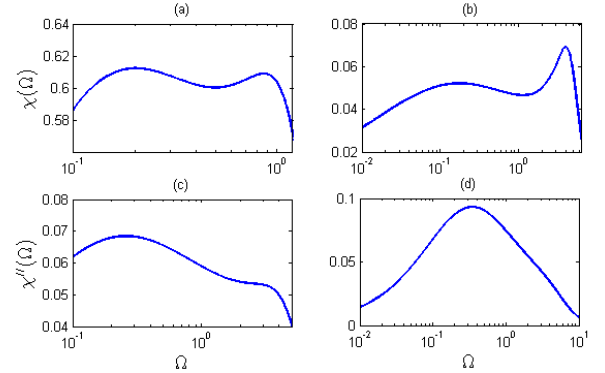


FIG. 12: The imaginary part of the complex susceptibility for various α . A double peak phenomena is observed. (a) $\alpha = 0.66$, $\gamma = 1.8$ and $\omega = 0.7$. (b) $\alpha = 0.5$, $\gamma = 10$ and $\omega = 2$. (c) $\alpha = 0.63$, $\gamma = 10$ and $\omega = 2$. (d) $\alpha = 0.8$, $\gamma = 10$ and $\omega = 2$.

monotonic decay phase for the correlation function. The reason is the same, the friction becomes more of an elastic force for such small α embedding oscillations in the system. Such a claim is emphasized by the Cole-Cole plots of the complex susceptibility, presented in Fig. 14. For $\alpha = 0.8$ the behavior is very much as for a normal damped oscillator, a Debye model [16], for small enough $\omega/\gamma^{1/(2-\alpha)}$ (large friction) - Fig. 14.(c), i.e. a monotonic behavior for the relaxation, and a Van Vleck-Weisskopf-Fröhlich type [16] for a large value of $\omega/\gamma^{1/(2-\alpha)}$ (small friction)- Fig. 14.(d), i.e. a oscillating behavior of the re-

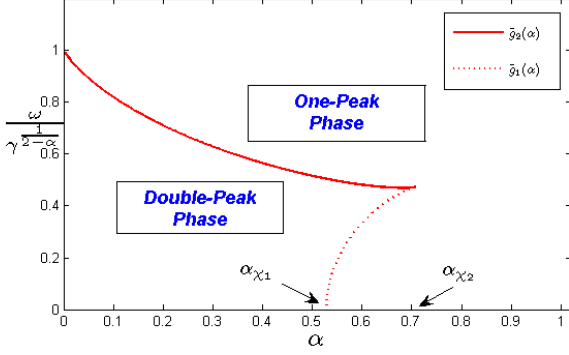


FIG. 13: Phase diagram of the imaginary part of complex susceptibility. Two phases are found, a phase with a presence of one maxima for $\chi''(\Omega)$ - “One-Peak” phase, or a phase with a presence of two maxima - “Double-Peak” phase. For $\alpha > \alpha_{\chi_2} = 0.707\dots$ only one phase is present.

laxation. These two prototypes of normal complex susceptibility correspond to presence of a single characteristic frequency in the system, or a single time scale if we are concerned with correlations. For small α , like $\alpha = 0.1$ in Fig. 14(a), we see a coexistence of these two types of normal susceptibilities. The right side of Fig. 14(a) corresponds to a Debye type - a monotonically decaying process and on the left side a Van Vleck-Weisskopf-Fröhlich type which shows highly oscillating behavior even in the case when γ and ω are the same as for Fig. 14(c). Effectively for small α we have two characteristic frequencies in the system, the lowest is responsible for the monotonic decay and the high frequency is an oscillating process, for intermediate α we have some mixed behavior - Fig. 14(b). This oscillating behavior that is seen in Fig. 14(a) is the manifestation of the cage effect which we already explained as the rattling motion of the surrounding particles and is presented in FLE due to the friction force.

VII. SUMMARY

The Fractional Langevin Equation (FLE) with power-law memory kernel and $0 < \alpha < 1$ is a stochastic framework describing anomalous sub-diffusive behavior. This equation may be expressed in terms of fractional derivative and so provides an example of a physical phenomena where non-integer calculus plays a central role. The solution of a fractional-differential equation describing the correlation function was presented in terms of roots of regular polynomials. It was shown that for $\alpha \neq 1$ there is no unique way to define an over-damped or under-damped motion. Three definitions were proposed for the frequency of transition, i.e. ω_c , ω_m and ω_z . We observed an existence of a phase transition for a critical $\alpha = \alpha_c \approx 0.402$, where for $\alpha < \alpha_c$ $C_x(t)$ does not decay

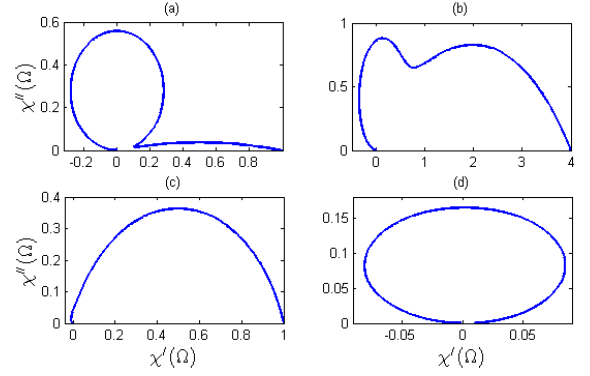


FIG. 14: The Cole-Cole plots of the complex susceptibility, $\chi''(\Omega)$ as a function of $\chi'(\Omega)$. (a) $\alpha = 0.1$, $\gamma = 10$ and $\omega = 1$. (b) $\alpha = 0.5$, $\gamma = 10$ and $\omega = 0.5$. (c) $\alpha = 0.8$, $\gamma = 10$ and $\omega = 1$. (d) $\alpha = 0.8$, $\gamma = 1$ and $\omega = 10$. Panel (c) and (d) show two typical normal behaviors, for small α the panels (a) and (b) show a behavior which is a mixture of the two normal typical behaviors.

monotonically for any $\omega > 0$. Physically it is explained using a cage effect: a rattling motion of a particle in the cage formed by the surrounding particles. A response to a time dependent field in terms of complex susceptibility $\chi(\Omega)$ also was calculated and similar critical α 's were found. Particularly for $\alpha < \alpha_R = 0.441\dots$ the system will always be in a resonance with the external field for particular Ω and any γ and ω , even in the case of free particle ($\omega = 0$). For the “loss” - the imaginary part of the complex susceptibility, $\chi''(\Omega)$, two phases were defined: (i) “One-Peak” phase where the complex susceptibility obtains only one maximum as in a regular case, (ii) “Double-Peak” phase, where the complex susceptibility obtains two maxima, a phase diagram was presented. Two critical exponents $\alpha_{\chi_1} = 0.527\dots$ and $\alpha_{\chi_2} = 0.707\dots$ were found for $\chi''(\Omega)$, exponents which define the boundaries of the phase diagram. In conclusion, critical exponents like α_c , α_R , α_{χ_1} and α_{χ_2} , mark sharp transitions in the behaviors of systems with fractional dynamics. Thus, these critical exponents are clearly important and general in the description of anomalous kinetics.

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VIII. APPENDIX A: THE SOLUTION FOR NON-DISTINCT ZEROS OF $\hat{P}(s)$

In this Appendix we derive the solution of Eq. (12) for the case when two zeros of $\hat{P}(s)$ coincide. This means that $\hat{P}(s)$ has $2q - 2$ distinct zeros of order 1 and one zero of order 2. Namely, at the critical point ω_c only two a_k coincide and we present here a method of solution for $\omega = \omega_c$.

Starting with Eq. (12), we write the partial fraction

expansion in the following way

$$\frac{1}{\hat{P}(s)} = \sum_{k=1}^{2q-1} \frac{A_k}{s - a_k} + \frac{\tilde{A}}{(s - a_{2q-1})^2} \quad (85)$$

where a_k are the zeros of $\hat{P}(s)$ and we assign a_{2q-1} to be the zero of the second order. A_k and \tilde{A} are given by

$$A_k = \frac{1}{\frac{d\hat{P}(s)}{ds} \big|_{s=a_k}} \quad 1 \leq k < 2q-1, \quad (86)$$

$$A_{2q-1} = - \sum_{k=1}^{2q-2} A_k, \quad (87)$$

and

$$\tilde{A} = \frac{1}{\frac{d}{ds} \frac{\hat{P}(s)}{s - a_{2q-1}} \big|_{s=a_{2q-1}}} \quad (88)$$

Using the relation [18]

$$\sum_{k=1}^{2q-1} a_k^m A_k + m a_k^{m-1} \tilde{A} = 0 \quad m = 0, 1, \dots, 2q-2 \quad (89)$$

one finds that

$$\frac{s^m}{\hat{P}(s)} = \sum_{k=1}^{2q-1} \frac{A_k a_k^m}{s - a_k} + \frac{m a_{2q-1}^{m-1} \tilde{A}}{s - a_{2q-1}} + \frac{a_{2q-1}^m \tilde{A}}{(s - a_{2q-1})^2} \quad m = 0, 1, \dots, 2q-1. \quad (90)$$

Hence using Eqs. (12,18,90)

$$\hat{C}_x(s) = \sum_{m=0}^{2q-1} \sum_{j=0}^{q-1} \left(\sum_{k=1}^{2q-1} \frac{a_k^m A_k \tilde{B}_{mj}}{s - a_k} s^{-\frac{j}{q}} + \frac{m a_{2q-1}^{m-1} \tilde{A} \tilde{B}_{mj}}{s - a_{2q-1}} s^{-\frac{j}{q}} + \frac{a_{2q-1}^m \tilde{A} \tilde{B}_{mj}}{(s - a_{2q-1})^2} s^{-\frac{j}{q}} \right), \quad (91)$$

and it is only left to perform an Inverse Laplace Transform of $\frac{1}{s^{\frac{j}{q}}(s - a_{2q-1})^2}$, using convolution theorem

$$\frac{1}{s^{\frac{j}{q}}(s - a_{2q-1})^2} \bullet \circ \frac{t e^{a_{2q-1}t}}{\Gamma(\frac{j}{q}) a_{2q-1}^{\frac{j}{q}}} \gamma(\frac{j}{q}, a_{2q-1}t) - \frac{e^{a_{2q-1}t}}{\Gamma(\frac{j}{q}) a_{2q-1}^{\frac{j}{q}+1}} \gamma(\frac{j}{q} + 1, a_{2q-1}t) \quad (92)$$

or using Mittag-Leffler function

$$\frac{1}{s^{\frac{j}{q}}(s - a_{2q-1})^2} \bullet \circ t^{\frac{j}{q}+1} E_{1,1+\frac{j}{q}}(a_{2q-1}t) - \frac{j}{q} t^{\frac{j}{q}} E_{1,2+\frac{j}{q}}(a_{2q-1}t). \quad (93)$$

Finally using Eqs. (21,91,92)

$$C_x(t) = \sum_{m=0}^{2q-1} \sum_{j=0}^{q-1} \frac{\tilde{B}_{mj}}{\Gamma(\frac{j}{q})} \left[\sum_{k=1}^{2q-1} a_k^{m-\frac{j}{q}} A_k e^{a_k t} \gamma(\frac{j}{q}, a_k t) + a_{2q-1}^{m-\frac{j}{q}} \tilde{A} e^{a_{2q-1}t} \left((t + m a_{2q-1}^{-1}) \gamma(\frac{j}{q}, a_{2q-1}t) - a_{2q-1}^{-1} \gamma(\frac{j}{q} + 1, a_{2q-1}t) \right) \right], \quad (94)$$

or using Eq. (93)

$$C_x(t) = \sum_{m=0}^{2q-1} \sum_{j=0}^{q-1} \tilde{B}_{mj} t^{\frac{j}{q}} \left[\sum_{k=1}^{2q-1} a_k^m A_k E_{1,1+\frac{j}{q}}(a_k t) + a_{2q-1}^m \tilde{A} \left((t + m a_{2q-1}^{-1}) E_{1,1+\frac{j}{q}}(a_{2q-1}t) - \frac{j}{q} t E_{1,2+\frac{j}{q}}(a_{2q-1}t) \right) \right]. \quad (95)$$

A final remark: one can show that for our case of integer $q > p > 0$, third and higher order zeros of $\hat{P}(s)$ don't exist.

IX. APPENDIX B: EXPLORATION OF EQ. (80)

In this Appendix we prove the existence of α_R for FLE with a harmonic force and derive the equation for $g(\alpha)$

given by Eq. (82). Solving Eq. (80) one gets for $\gamma/\Omega_R^{2-\alpha}$

$$\frac{\gamma}{\Omega_R^{2-\alpha}} = \frac{1}{2\alpha} \left([2 - \alpha y] \cos\left(\frac{\pi\alpha}{2}\right) \right) \pm \frac{1}{2\alpha} \sqrt{\left([2 - \alpha y]^2 \cos^2\left(\frac{\pi\alpha}{2}\right) + 8\alpha y \right)} = q_{1\pm}(y), \quad (96)$$

and $y = \left(\frac{\omega^2}{\Omega_R^2} - 1 \right) > -1$. Writing the left hand side of Eq. (96) in terms of y

$$\frac{\gamma}{\Omega_R^{2-\alpha}} = \frac{\gamma}{\omega^{2-\alpha}} (y+1)^{1-\frac{\alpha}{2}} = q_2(y). \quad (97)$$

We see that for the extrema points of $R(\Omega)$ the functions $q_{1\pm}(y)$ and $q_2(y)$ cross each other (see Fig. 15). While $q_2(y)$ is a monotonic increasing function starting from zero for $y = -1$ and growing as $y^{1-\frac{\alpha}{2}}$ for large y , $q_{1\pm}(y)$ constructs two branches where $q_{1+}(y)$ is the upper branch and $q_{1-}(y)$ is the lower branch and for some point y_α

$$q_{1-}(y_\alpha) = q_{1+}(y_\alpha) = \frac{(2 - \alpha y_\alpha) \cos\left(\frac{\pi\alpha}{2}\right)}{2\alpha}. \quad (98)$$

If $y_\alpha < -1$ then $q_2(y)$ crosses $q_{1\pm}(y)$ no matter what the parameters γ, ω and α are, because in that case for $y = -1$ $q_{1\pm}(y) > 0$ and $q_2(y) = 0$ and a resonance is always obtained. The point y_α is derived from Eq. (96) and determined by the following relation

$$(2 - \alpha y_\alpha)^2 \cos^2\left(\frac{\pi\alpha}{2}\right) + 8\alpha y_\alpha = 0. \quad (99)$$

Solving Eq. (99) in terms of y_α one finds

$$y_\alpha = -\frac{2}{\alpha \cos^2\left(\frac{\pi\alpha}{2}\right)} \left(1 - \sin\left(\frac{\pi\alpha}{2}\right) \right)^2, \quad (100)$$

where we took the $-$ sign because $y > -1$. For $0 < \alpha < 1$ Eq. (100) is an increasing function of α and so we have a critical α , α_R for $y_\alpha = -1$. Eq. (99) with $y_\alpha = -1$ is exactly Eq. (77) which defines the equation for α_R and so we have shown the existence of α_R for the harmonically bounded particle.

We now argue that the boundary between “Resonance-Phase” and “No-Resonance Phase” is given by the following relation

$$\frac{\omega}{\gamma^{\frac{1}{2-\alpha}}} = g(\alpha), \quad (101)$$

where $g(\alpha)$ is only α dependent and equals zero for $\alpha \leq \alpha_R$. One readily sees from Eq. (96) that the large y behavior of $q_{1\pm}(y)$ is proportional to $\pm\sqrt{y}$, where $q_2(y)$ behaves like $y^{1-\frac{\alpha}{2}}$ for large y (Eq. (97)), also we note that $\frac{dq_{1\pm}(y)}{dy}$ and $\frac{dq_2(y)}{dy}$ are monotonically decaying functions of y . As a result we have four options for the scenario of $q_2(y)$ crossing $q_{1\pm}(y)$ (see also Fig. 15):

(i) $q_2(y_1) = q_{1-}(y_1)$ and $q_2(y_2) = q_{1+}(y_2)$ for $y_1 < y_2$,

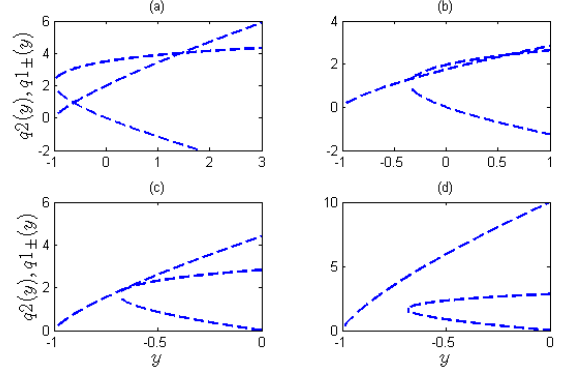


FIG. 15: The four scenarios for $q_{1\pm}(y)$ and $q_2(y)$ crossings. Existence of a crossings correspond to “Resonance-Phase” and no crossings correspond to “No-Resonance Phase”. (a) $\alpha = 0.441$, $\gamma = 2$ and $\omega = 1$, for this case there will always be 2 crossings, $\alpha < \alpha_R$. (b) $\alpha = 0.6$, $\gamma = 1.75$ and $\omega = 1$, this panel corresponds to a “Resonance-Phase” (c) $\alpha = 0.5$, $\gamma = 4.4$ and $\omega = 1$, this panel describes a situation on the boundary between “Resonance-Phase” and “No-Resonance Phase”. (d) $\alpha = 0.5$, $\gamma = 10$ and $\omega = 1$, a “No-Resonance Phase”.

- (ii) $q_2(y_1) = q_{1+}(y_1)$ and $q_2(y_2) = q_{1+}(y_2)$ for $y_1 < y_2$,
- (iii) $q_2(y) = q_{1+}(y)$ for a single y
- (iv) $q_2(y) \neq q_{1+}(y)$ and $q_2(y) \neq q_{1-}(y)$ for any y .

When there are two crossings then the one with the larger y corresponds to minimum and the smaller y corresponds to maximum and belongs to the “Resonance-Phase”. When there are no crossings then we are in the “No-Resonance Phase”. The scenario (iii) corresponds exactly to the boundary between the two phases. In order to find the boundary, two conditions must be fulfilled

$$q_2(y_1) = q_{1+}(y_1) \quad (102)$$

and

$$\frac{dq_2(y)}{dy} \Big|_{y=y_1} = \frac{dq_{1+}(y)}{dy} \Big|_{y=y_1}, \quad (103)$$

as illustrated in panel (c) of Fig. 15. Starting from Eq. (102) we compare the left-hand side to some constant c and using Eq. (97) we find

$$y = c^{\frac{2}{2-\alpha}} \left(\frac{\omega}{\gamma^{\frac{1}{2-\alpha}}} \right)^2 - 1. \quad (104)$$

Comparing the right hand-side of Eq. (102) to the same c and using Eq. (104) we arrive to the following relation

$$\left(\frac{\omega}{\gamma^{\frac{1}{2-\alpha}}} \right)^2 = \frac{c^2 - c \cos\left(\frac{\pi\alpha}{2}\right) \left(1 + \frac{2}{\alpha}\right) + \frac{2}{\alpha} c^{-\frac{2}{2-\alpha}}}{\frac{2}{\alpha} - c \cos\left(\frac{\pi\alpha}{2}\right)}. \quad (105)$$

Next performing the derivation in Eq. (103) and using Eq. (104) we find

$$\left(\frac{\omega}{\gamma^{\frac{1}{2-\alpha}}}\right)^2 = \frac{(2-\alpha)c(2c - (1 + \frac{2}{\alpha}\cos(\frac{\pi\alpha}{2})))}{\frac{4}{\alpha} - 4c\cos(\frac{\pi\alpha}{2}) + \alpha c\cos(\frac{\pi\alpha}{2})} c^{-\frac{2}{2-\alpha}}, \quad (106)$$

comparison of Eq. (106) and Eq. (105) supplies an equation for c

$$\alpha^3 \cos\left(\frac{\pi\alpha}{2}\right) c^3 + (2\alpha - 5\alpha^2 - \alpha(2 + \alpha)\cos(\pi\alpha)) c^2 + 12\alpha \cos\left(\frac{\pi\alpha}{2}\right) c - 8 = 0. \quad (107)$$

Eq. (107) has three different solutions where only one is real for $0 < \alpha < 1$, we will call it $c(\alpha)$ and

$c(\alpha) =$

$$\begin{aligned} & \frac{1}{3\alpha^3} \left[\sec\left(\frac{\pi\alpha}{2}\right) \left\{ -2\alpha + 5\alpha^2 + 2\alpha\cos(\pi\alpha) + \alpha^2\cos(\pi\alpha) \right. \right. \\ & \left. \left. - \left(2^{\frac{5}{3}}\alpha^2\sin^2\left(\frac{\pi\alpha}{2}\right) (-4 + 20\alpha - 7\alpha^2 + (2 + \alpha)^2\cos(\pi\alpha))\right) \right. \right. \\ & \left. \left. [-80\alpha^3 + 312\alpha^4 - 384\alpha^5 + 152\alpha^6 - 3\alpha^3(-40 + 132\alpha - 126\alpha^2 + 43\alpha^3)\cos(\pi\alpha) - 48\alpha^3\cos(2\pi\alpha) + 72\alpha^4\cos(2\pi\alpha) \right. \right. \\ & \left. \left. - 24\alpha^6\cos(2\pi\alpha) + 8\alpha^3\cos(3\pi\alpha) + 12\alpha^4\cos(3\pi\alpha) + 6\alpha^5\cos(3\pi\alpha) + \alpha^6\cos(3\pi\alpha) \right. \right. \\ & \left. \left. 24\sqrt{6}\sqrt{-\alpha^9\cos^2\left(\frac{\pi\alpha}{2}\right) \left(32 - 204\alpha + 204\alpha^2 - 59\alpha^3 + (2 + \alpha)^2(-8 + 5\alpha)\cos(\pi\alpha)\sin^4\left(\frac{\pi\alpha}{2}\right)\right)} \right]^{-\frac{1}{3}} \right. \\ & \left. + \frac{1}{2^{\frac{1}{3}}} \left([-80\alpha^3 + 312\alpha^4 - 384\alpha^5 + 152\alpha^6 - 3\alpha^3(-40 + 132\alpha - 126\alpha^2 + 43\alpha^3)\cos(\pi\alpha) - 48\alpha^3\cos(2\pi\alpha) + 72\alpha^4\cos(2\pi\alpha) \right. \right. \\ & \left. \left. - 24\alpha^6\cos(2\pi\alpha) + 8\alpha^3\cos(3\pi\alpha) + 12\alpha^4\cos(3\pi\alpha) + 6\alpha^5\cos(3\pi\alpha) + \alpha^6\cos(3\pi\alpha) \right. \right. \\ & \left. \left. 24\sqrt{6}\sqrt{-\alpha^9\cos^2\left(\frac{\pi\alpha}{2}\right) \left(32 - 204\alpha + 204\alpha^2 - 59\alpha^3 + (2 + \alpha)^2(-8 + 5\alpha)\cos(\pi\alpha)\sin^4\left(\frac{\pi\alpha}{2}\right)\right)} \right]^{-\frac{1}{3}} \right) \right] \Bigg\}. \end{aligned} \quad (108)$$

We thus justified the use of Eq. (101) and $g(\alpha)$ is given by Eq. (82).

X. APPENDIX C: EXPLORATION OF $d\chi''(\Omega)/d\Omega$

We start with the exploration of $d\chi''(\Omega)/d\Omega$, where $\chi''(\Omega)$ is given by Eq. (84),

$$\begin{aligned} \frac{d\chi''(\Omega)}{d\Omega} = & \frac{\gamma\Omega^{\alpha+3}\sin\left(\frac{\pi\alpha}{2}\right)}{\left[(\omega^2 - \Omega^2)^2 + \gamma^2\Omega^{2\alpha} + 2\gamma(\omega^2 - \Omega^2)\Omega^\alpha\cos\left(\frac{\pi\alpha}{2}\right)\right]^2} \left[\alpha \left\{ \left(\frac{\omega^2}{\Omega^2} - 1\right)^2 + \left(\frac{\gamma}{\Omega^{2-\alpha}}\right)^2 + \right. \right. \\ & \left. \left. 2\left(\frac{\omega^2}{\Omega^2} - 1\right)\left(\frac{\gamma}{\Omega^{2-\alpha}}\right)\cos\left(\frac{\pi\alpha}{2}\right) \right\} - \left\{ -4\left(\frac{\omega^2}{\Omega^2} - 1\right) + 2\alpha\left(\frac{\gamma}{\Omega^{2-\alpha}}\right)^2 + 2\left(\alpha\left(\frac{\omega^2}{\Omega^2} - 1\right) - 2\right)\left(\frac{\gamma}{\Omega^{2-\alpha}}\right)\cos\left(\frac{\pi\alpha}{2}\right) \right\} \right], \end{aligned} \quad (109)$$

and we easily see that in order to $\frac{d\chi''(\Omega)}{d\Omega} = 0$, the following condition must be fulfilled

$$\alpha\left(\frac{\gamma}{\Omega^{2-\alpha}}\right)^2 - 4\left(\frac{\gamma}{\Omega^{2-\alpha}}\right)\cos\left(\frac{\pi\alpha}{2}\right) - (4y + \alpha y^2) = 0, \quad (110)$$

where $y = \left(\frac{\omega^2}{\Omega^2} - 1\right) > -1$. The left hand side of Eq. (110) is a second order polynomial in terms of $\frac{\gamma}{\Omega^{2-\alpha}}$,

which is easily solved

$$\left(\frac{\gamma}{\Omega^{2-\alpha}}\right) = \frac{4\cos\left(\frac{\pi\alpha}{2}\right)}{2\alpha} \pm \frac{1}{2\alpha} \sqrt{16\cos^2\left(\frac{\pi\alpha}{2}\right) + 4\alpha(4y + \alpha y^2)}. \quad (111)$$

The right hand side of Eq. (111) we will call $\tilde{q}_{1\pm}(y)$ and the left hand side $\tilde{q}_2(y)$,

$$\tilde{q}_2(y) = \frac{\gamma}{\omega^{2-\alpha}}(y + 1)^{1-\frac{\alpha}{2}}. \quad (112)$$

The crossings of $\tilde{q}_{1\pm}(y)$ and $\tilde{q}_2(y)$ determine the extrema points of $\chi''(\Omega)$ and using the fact that for $y \rightarrow \infty$ $\tilde{q}_2(y) \propto y^{1-\frac{\alpha}{2}}$ and $\tilde{q}_{1\pm}(y) \propto y$ we have six different scenarios for the crossing of $\tilde{q}_2(y)$ and $\tilde{q}_{1\pm}(y)$.

- (i) $\tilde{q}_2(y) = \tilde{q}_{1-}(y)$ for a single y and there are no other crossings,
- (ii) $\tilde{q}_2(y_1) = \tilde{q}_{1-}(y_1)$ and $\tilde{q}_2(y_2) = \tilde{q}_{1+}(y_2)$ for $y_1 < y_2$,
- (iii) $\tilde{q}_2(y_1) = \tilde{q}_{1-}(y_1)$, $\tilde{q}_2(y_2) = \tilde{q}_{1+}(y_2)$ and $\tilde{q}_2(y_3) = \tilde{q}_{1+}(y_3)$ for $y_1 < y_2 < y_3$,
- (iv) $\tilde{q}_2(y_1) = \tilde{q}_{1+}(y_1)$, $\tilde{q}_2(y_2) = \tilde{q}_{1+}(y_2)$ and $\tilde{q}_2(y_3) = \tilde{q}_{1+}(y_3)$ for $y_1 < y_2 < y_3$,
- (v) $\tilde{q}_2(y_1) = \tilde{q}_{1+}(y_1)$ and $\tilde{q}_2(y_2) = \tilde{q}_{1+}(y_2)$ for $y_1 < y_2$,
- (vi) $\tilde{q}_2(y) = \tilde{q}_{1+}(y)$ for a single y and there are no other crossings.

Generally if there is only one crossing, scenario (i) and (vi), the meaning is that $\chi''(\Omega)$ will have only one maximum and on the contrary when there are three crossings, scenario (iii) and (iv), there are two maximums and one minimum for $\chi''(\Omega)$. These correspond to two different phases the “One-Peak” phase and the “Double-Peak” phase where the scenarios (ii) and (v) are the boundaries between these phases. We are interested in finding these boundaries, where for scenario (ii) and (v) two conditions must be fulfilled

$$\tilde{q}_2(y_1) = \tilde{q}_{1+}(y_1) \quad (113)$$

and

$$\left. \frac{d\tilde{q}_2(y)}{dy} \right|_{y=y_1} = \left. \frac{d\tilde{q}_{1+}(y)}{dy} \right|_{y=y_1}. \quad (114)$$

Starting from Eq. (113) we compare the left-hand side to some constant \tilde{c} and using Eq. (112) we find

$$y = \tilde{c}^{\frac{2}{2-\alpha}} \frac{\omega^2}{\gamma^{\frac{2}{2-\alpha}}} - 1. \quad (115)$$

Comparing the right hand side of Eq. (113) to the same \tilde{c} and using Eq. (115) we arrive to the following relation

$$\left(\frac{\omega}{\gamma^{\frac{1}{2-\alpha}}} \right)^2 = \left(1 - \frac{2}{\alpha} + \frac{1}{2} \sqrt{\frac{16}{\alpha^2} - 4 \left(\frac{4 \cos(\frac{\pi\alpha}{2})}{\alpha} \tilde{c} - \tilde{c}^2 \right)} \right) \tilde{c}^{-\frac{2}{2-\alpha}}. \quad (116)$$

Next performing the derivation in Eq. (114) and using Eq. (115) we find

$$\left(\frac{\omega}{\gamma^{\frac{1}{2-\alpha}}} \right)^2 = \left(\frac{1}{2} - \frac{1}{\alpha} + \frac{1}{2} \sqrt{\frac{(4-2\alpha)^2}{4\alpha^2} + \tilde{c} \left(\frac{2-\alpha}{\alpha} \right) (2\alpha\tilde{c} - 4 \cos(\frac{\pi\alpha}{2}))} \right) \tilde{c}^{-\frac{2}{2-\alpha}}, \quad (117)$$

comparison of Eq. (116) and Eq. (117) supplies an equation for \tilde{c}

$$4\alpha^2\tilde{c}^4 - 16 \cos\left(\frac{\pi\alpha}{2}\right) (2+\alpha)\tilde{c}^3 + \frac{16}{\alpha^2} (-4 + 8\alpha - \alpha^2 + \cos^2\left(\frac{\pi\alpha}{2}\right) (2+\alpha)^2) \tilde{c}^2 - \frac{64}{\alpha^2} \cos\left(\frac{\pi\alpha}{2}\right) (6-\alpha)\tilde{c} + \frac{64}{\alpha^3} (4-\alpha) = 0. \quad (118)$$

Eq. (118) is a forth order polynomial and could be solved by standard methods or using Mathematica. It has 4 different solutions while two of the solutions have non-zero Imaginary parts for any $0 < \alpha < 1$, while the other two solutions have no Imaginary part for $\alpha < \alpha_{\chi_2} \approx 0.70776$. Lets call these solutions $\tilde{c}_1(\alpha)$ and $\tilde{c}_2(\alpha)$, the non-zero Imaginary part for $\alpha > \alpha_{\chi_2}$ of both $\tilde{c}_1(\alpha)$ and $\tilde{c}_2(\alpha)$ means that only scenario (vi) is applicable for such α 's and we are in the “One-Peak” phase. The boundaries between the two phases are given by

$$\frac{\omega}{\gamma^{\frac{1}{2-\alpha}}} = \tilde{g}_{1,2}(\alpha) = \sqrt{1 - \frac{2}{\alpha} + \frac{1}{2} \sqrt{\frac{16}{\alpha^2} - 4 \left(\frac{4 \cos(\frac{\pi\alpha}{2})}{\alpha} \tilde{c}_{1,2}(\alpha) - \tilde{c}_{1,2}^2(\alpha) \right)}} \tilde{c}_{1,2}^{-\frac{1}{2-\alpha}}(\alpha) \quad (119)$$

where the subscript 1 is for $\tilde{g}_1(\alpha)$ the lower bound in Fig. 13 and subscript 2 is the upper bound $\tilde{g}_2(\alpha)$ in Fig. 13. For $\tilde{g}_1(\alpha)$ there is also another interesting point $\alpha_{\chi_1} = 0.527031$ which satisfies the following relation

$$\alpha_{\chi_1}^2 - 4\alpha_{\chi_1} + 4 \cos^2\left(\frac{\pi\alpha_{\chi_1}}{2}\right) = 0, \quad (120)$$

for $\alpha < \alpha_{\chi_1}$ $\tilde{g}_1(\alpha) = 0$.

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